TROPICAL ADDITIVE SYNTHESIS

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Abstract. We introduce Tropical Additive Synthesis, a new, and efficient, synthesis based on the theory of Tropical Mathematics. We developed the tropical version of the concepts of usual Additive Synthesis, as beating and classical waveforms. Finally we describe the architecture of the Tropical Oscillator.

1. Introduction

Tropical mathematics is an exciting new field based on the so-called *min-plus algebra* with several applications in applied mathematics, including control theory, optimization and mathematical physics ([2, 3, 7]).

The adjective "tropical" was coined by French mathematicians, including Jean-Eric Pin [7], in the honor of their Brazilian colleague Imre Simon [9], who was one of the pioneers in min-plus algebra. There is no deeper meaning in the adjective "tropical". It simply stands for the French view of Brazil.

The concept of "tropicalization" consists to replace our usual addition and multiplication with the tropical ones. We talk, for example, of Tropical Geometry, when we "tropicalize" objects (such as varieties and polynomials) in Algebraic Geometry ([6, 4]).

The idea of this paper is to apply the tropicalization process to concepts in Sound Synthesis. Not every concept can be tropicalized or, eventually, tropicalization will lead to something not so useful or not so new. For example, as will be clearer later, the tropicalization of the ring modulation of two waveforms is simply their sum. In general, every process in sound synthesis involving products of waveforms will reduce, after tropicalization, to the sum of such waveforms.

Instead, processes involving sum of waveforms will lead to processes involving the minimum of these waveforms, an operation less common in sound synthesis. Thus, a first natural target of the tropicalization process is Additive Synthesis, leading to something that we call *Tropical Additive Synthesis*.

Tropical Additive Synthesis, as we will see in the next sections, permits to create complex waveforms, starting from few sine waves, as in the usual Additive Synthesis, but with more rich spectra.

The various basic aspects of Tropical Additive Synthesis will be treated in the related Section and its subsections. Then, in the next section, we pass to describe the architecture of the so-called $Trop(x[n])-$oscillator. The first section is an overview of the main facts in Tropical Mathematics.

2. Tropical Mathematics

We start by considering the so-called *min-plus algebra* $\mathbb{R}_{mp} =< \mathbb{T}, \oplus, \odot >$ which is an algebra, in particular a semi-ring, over the set $\mathbb{T} := \mathbb{R} \cup \{\infty\}$ and endowed by two binary operations $\oplus$ and $\odot$:

$$a \oplus b := \min\{a, b\} \qquad \text{e} \qquad a \odot b := a + b.$$

So, for example, one has $3 \oplus 4 = 3$ and $3 \odot 4 = 7$.

The mathematics performed in this algebra is usually called *Tropical Mathematics* and the operations $\oplus$ and $\odot$ are called respectively *tropical addition* and *tropical multiplication*.

Remark 1. In literature there is also a definition of tropical addition in terms of the maximum of two numbers, i.e. $a \oplus b := \max\{a, b\}$ and $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$. Clearly the two algebras are isomorphic under the map $a \mapsto -a$.

Many of the axioms of classical mathematics still holds in Tropical Mathematics. For example, both addition and multiplication are commutative and associative:

$$x \oplus y = y \oplus x \qquad x \odot y = y \odot x.$$

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z \qquad x \odot (y \odot z) = (x \odot y) \odot z.$$

Moreover the distributivity of multiplication with respect to addition holds:

$$x \odot (y \oplus z) = x \odot y \oplus x \odot z.$$

Both operations have a neutral element. The infinity, ∞, is the neutral element of the addition and 0 is the neutral element of the multiplication:

$$x \oplus \infty = x, \quad x \odot 0 = x.$$

Moreover the following identities hold in $\mathbb{R}_{mp}$

$$x \odot \infty = \infty, \quad \infty \odot \infty = \infty. \tag{2.1}$$

Another identity, regarding the 0 element, is the following one:

$$x \oplus 0 = \begin{cases} 0 & \text{if } x \geq 0 \\ x & \text{if } x \leq 0 \end{cases}.$$

Tropical arithmetics appears simpler than the classical one. For example, in the study of the tables of operations, less effort is required (if compared with the ones in classical arithmetics):

$\oplus$	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2
3	1	2	3	3	3	3	3
4	1	2	3	4	4	4	4
5	1	2	3	4	5	5	5
6	1	2	3	4	5	6	6
7	1	2	3	4	5	6	7

$\odot$	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	2	3	4	5	6	7	8
2	3	4	5	6	7	8	9
3	4	5	6	7	8	9	10
4	5	6	7	8	9	10	11
5	6	7	8	9	10	11	12
6	7	8	9	10	11	12	13
7	8	9	10	11	12	13	14

Remark 2. Consider the equation $3 \oplus x = 10$. By the definition of $\oplus$, the equation does not admit any solution in $\mathbb{R}_{mp}$, that is, there is not a tropical subtraction $10 \ominus 3$. Hence the notion of subtraction is not compatible with the algebraic structure of $\mathbb{R}_{mp}$. However, the classical subtraction has meaning in the tropical semi-ring since it coincides with the tropical division: $a \oslash b = a - b$.

Proposition 1. The following properties holds

(i) $x \oplus x = x$, $\forall x \in \mathbb{T}$, that is every element in $\mathbb{R}_{trop}$ is idempotent.
(ii) $\forall x, y \in \mathbb{T}$ and $\forall n \in \mathbb{N}$ one has $(x \oplus y)^{\odot n} = x^{\odot n} \oplus y^{\odot n}$ where $x^{\odot n} = \underbrace{x \odot x \odot \cdots \odot x}_{n \text{ times}}$.

Property (*ii*) of Proposition 1 is known as "Freshman's Dream". The interested reader con find more about Tropical Geometry in [5].

3. Tropical Additive Synthesis

The basic idea of Tropical Additive Synthesis is to replace sum and multiplication with the correspondent tropical operations in the following Fourier Series:

$$x[n] = a_1 \cos(\omega_1 n + \phi_1) + a_2 \cos(\omega_2 n + \phi_2) + \cdots + a_p \cos(\omega_p n + \phi_p). \tag{3.1}$$

where each $a_i \cos(\omega_i n + \phi_i)$ represents a sine wave of angular frequency ω_i, phase ϕ_i and amplitude a_i ([8]). We suppose $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_p$. Notice that if $\omega_i = i\omega_1$, for $i = 2, \ldots, p$, we obtain a harmonic sound. Tropicalizing the previous formula we get

$$\begin{aligned} trop(x[n]) &= a_1 \odot \cos(\omega_1 n + \phi_1) \oplus a_2 \odot \cos(\omega_2 n + \phi_2) \oplus \cdots \oplus a_p \odot \cos(\omega_p n + \phi_p) \\ &= \min\{a_1 + \cos(\omega_1 n + \phi_1), a_2 + \cos(\omega_2 n + \phi_2), \ldots, a_p + \cos(\omega_p n + \phi_p)\} \end{aligned} \tag{3.2}$$

The careful reader may ask why we do not tropicalize inside the argument of each cosine. Observe that tropicalizing the argument of the cosine in $y[n] = a\cos(\omega n + \phi)$ we get

$$trop(y[n]) = a \odot \cos(\omega \odot n \oplus \phi) = a + \cos(\min\{\omega + n, \phi\}).$$

It is clear that when $n \geq \phi - \omega$ the previous term reduces to $trop(y[n]) = a + \cos(\phi)$ giving a sine wave with zero angular frequency (Figure 1).

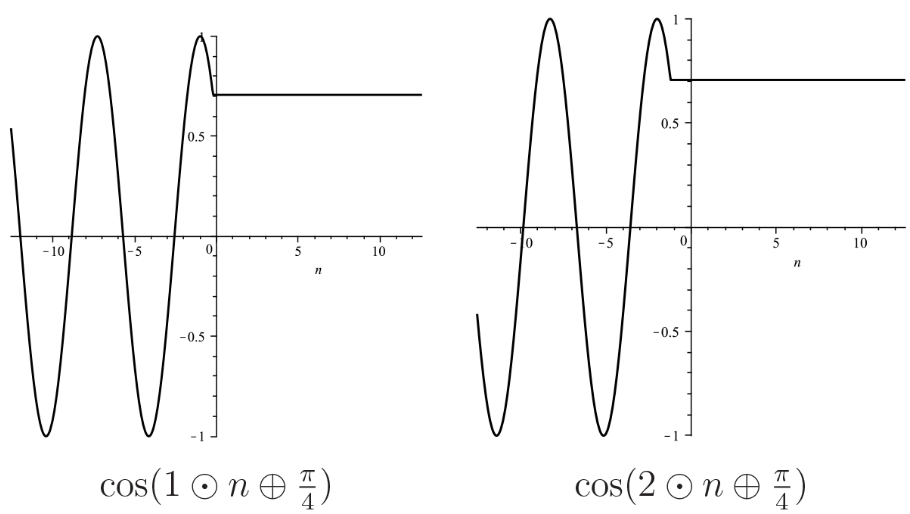

Figure 1. The effects of tropicalizing the argument of a cosine.

Hence, if we "fully" tropicalize the series in (3.1) we would get

$$trop(x[n]) = a_1 \odot \cos(\omega_1 \odot n \oplus \phi_1) \oplus \cdots \oplus a_p \odot \cos(\odot\omega_p \odot n \oplus \phi_p)$$

Here, for n large enough, each cosine will become constant, and $trop(x[n])$ will be the minimum of a set of fixed numbers (each of them given by the constant value of the cosine plus its amplitude). For example, if

$$trop(x[n]) = \cos(1 \odot n \oplus \frac{\pi}{4}) \oplus \cos(2 \odot n \oplus \frac{\pi}{3}) \oplus \cos(3 \odot n \oplus \frac{\pi}{6}) \oplus \cos(4 \odot n \oplus \frac{\pi}{4})$$

then the first cosine will be constant for $n \geq \frac{\pi}{4} - 1$, the second cosine will be constant for $n \geq \frac{\pi}{3} - 2$, the third cosine will be constant for $n \geq \frac{\pi}{6} - 3$ and the fourth cosine will be constant for $n \geq \frac{\pi}{4} - 4$. Hence, for $n \geq \max\{\frac{\pi}{4} - 1, \frac{\pi}{3} - 2, \frac{\pi}{6} - 3, \frac{\pi}{4} - 4\} = \frac{\pi}{4} - 1$, $trop(x[n])$ reduces to

$$trop(x[n]) = \cos(\frac{\pi}{4}) \oplus \cos(\frac{\pi}{3}) \oplus \cos(\frac{\pi}{6}) \oplus \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} \oplus \frac{\sqrt{3}}{2} \oplus \frac{1}{2} \oplus \frac{\sqrt{2}}{2} = \frac{1}{2}$$

as we can see in Figure 2.

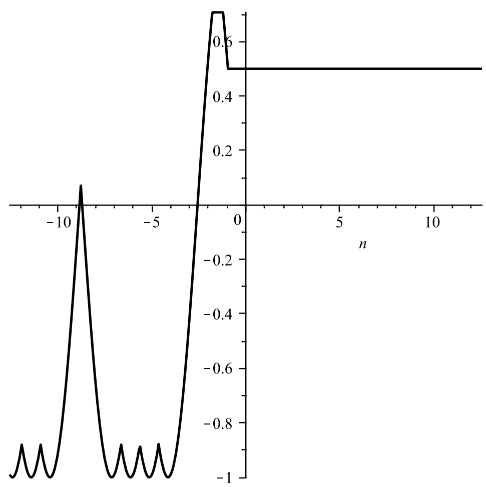

FIGURE 2. Graph of $\cos(1 \odot n \oplus \frac{\pi}{4}) \oplus \cos(2 \odot n \oplus \frac{\pi}{3}) \oplus \cos(3 \odot n \oplus \frac{\pi}{6}) \oplus \cos(4 \odot n \oplus \frac{\pi}{4})$.

Let us come back to Formula (3.2) and start to analyze the behavior of $trop(x[n])$. First of all we notice that, due to tropical multiplication, the amplitudes are now added to the cosine function, resulting in a translation on the $y-$axes of the cosine functions. It easy to see that $-1 + a_i \leq a_i \odot \cos(\omega_i n + \phi_i) \leq 1 + a_i$ and we have the following result.

Proposition 2 Given $x[n]$ as in (3.1), let $\alpha = \min\{a_1, a_2, \ldots, a_p\}$, then

$$(i) \quad \min(trop(x[n])) = 1 - \alpha$$

$$(ii) \quad \max(trop(x[n])) \leq 1 + \alpha$$

Proof. Since $1 - \alpha$ is achieved by one of the cosine function in (3.2), this minimum is achieved by the whole function $trop(x[n])$, by definition of the tropical addition, hence (i) is verified. For (ii) notice that there is at least one cosine function with reaches the value $1 + \alpha$, but the whole function $trop(x[n])$ can reach this value only if one or more terms reach it in a sample number n and all other terms have greater values in n. In this case we get equality in (ii), otherwise the maximum value of $trop(x[n])$ will be lower, but limited by $1 + \alpha$, giving the inequality in (ii). □

Remark 3. From the proposition is clear that the tropical wave $trop(x[n])$ varies between $1+\alpha$ e $-1-\alpha$. Hence, if we want to return to the usual range for waves, we need to add a dc offset, which, in the tropical setting, is obtained by tropically multiplying $trop(x[n])$ by a suitable constant term. The best choice is surely $-\alpha$, giving $-\alpha \odot trop(x[n]) = -\alpha + trop(x[n])$, but other values can be chosen to get different sound results.

To visualize the graph of $trop(x[n])$ we can proceed in the following way: we first drawn the graphs of all terms $a_i \odot \cos(\omega_i n + \phi_i)$ then, in each sample n we consider the lower point among all terms. The curve obtained by all these lower points is the graph of $trop(x[n])$. We refer to this curve as the *lower graph* of the terms $a_i \odot \cos(\omega_i n + \phi_i)$, $i = 1, \dots, p$. In Figure 3 we can see the graph of

$$trop(x[n]) = 1 \odot \cos(n) \oplus 0.5 \odot \cos(2.5n + \frac{\pi}{4}) \oplus 0.7 \odot \cos(4.2n + \frac{\pi}{2}) \oplus 1 \odot \cos(10n + \frac{\pi}{3})$$

as the lower graph of the terms $1 \odot \cos(n)$, $0.5 \odot \cos(2.5n + \frac{\pi}{4})$, $0.7 \odot \cos(4.2n + \frac{\pi}{2})$ and $1 \odot \cos(10n + \frac{\pi}{3})$. Here $\alpha = \min\{1, 0.5, 0.7, 1\} = 0.5$ and the minimum $-0.5 = -1 + \alpha$ is achieved three times by the dashed

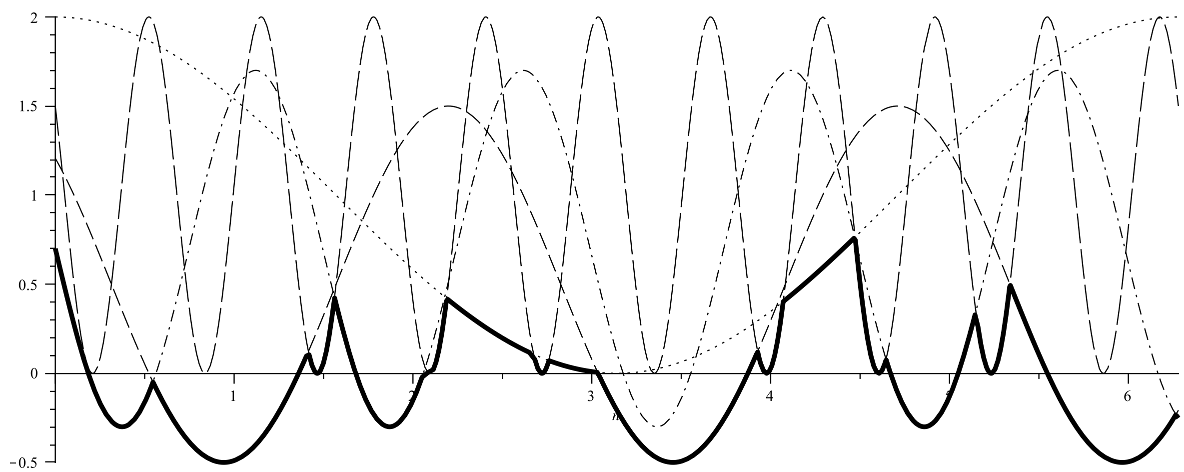

FIGURE 3. Dotted curve: $1 \odot \cos(n)$; dashed curve: $0.5 \odot \cos(2.5n + \frac{\pi}{4})$; dash-doted curve: $0.7 \odot \cos(4.2n + \frac{\pi}{2})$; long-dashed curve: $1 \odot \cos(10n + \frac{\pi}{3})$; solid curve: $trop(x[n])$.

curve representing $0.5 \odot \cos(2.5n + \frac{\pi}{4})$. Instead $trop(x[n])$ never reaches the maximum $1 + \alpha = 1.5$ of the dashed curve, since for the corresponding values of n for which $0.5 \odot \cos(2.5n + \frac{\pi}{4})$ is at the maximum, some other curves are below it so giving a smaller value for the lover graph.

Let us make now some considerations about the number of minimums of $trop(x[n])$. Clearly if the term $a_i \odot \cos(\omega_i n + \phi_i)$ has the lower amplitude, i.e $\alpha = a_i$, then each minimum of $a_i \odot \cos(\omega_i n + \phi_i)$ is a minimum of $trop(x[n])$. If there are other terms $a_j \odot \cos(\omega_j n + \phi_j)$ for which $\alpha = a_j$, then also their minimums will be minimums of $trop(x[n])$. To see this, compare the graph of Figure 3 with the one of Figure 4: here we have the same four terms, but with different amplitudes:

$$trop(x[n]) = 1 \odot \cos(n) \oplus 0.5 \odot \cos(2.5n + \frac{\pi}{4}) \oplus 0.5 \odot \cos(4.2n + \frac{\pi}{2}) \oplus 0.5 \odot \cos(10n + \frac{\pi}{3})$$

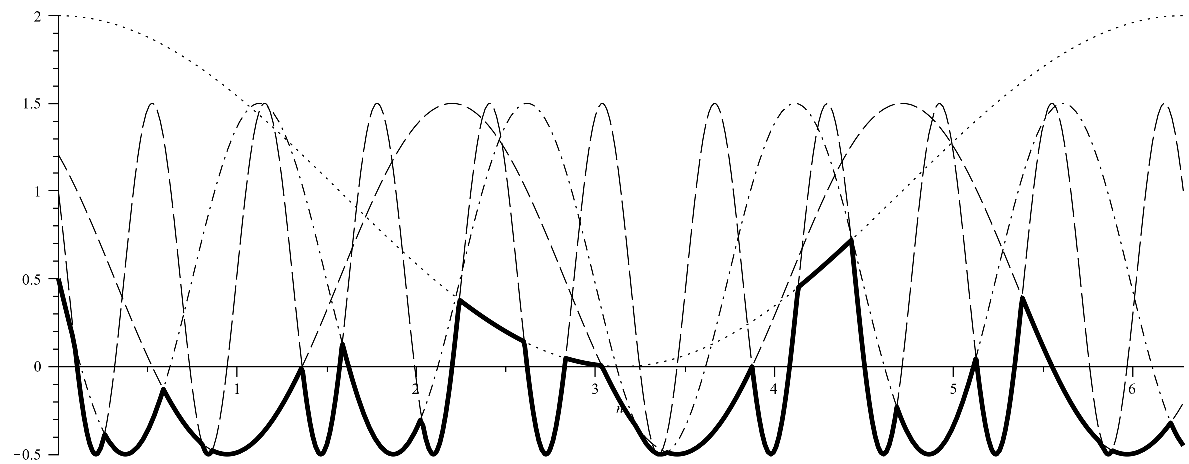

FIGURE 4. Dotted curve: $1 \odot \cos(n)$; dashed curve: $0.5 \odot \cos(2.5n + \frac{\pi}{4})$; dash-dotted curve: $0.5 \odot \cos(4.2n + \frac{\pi}{2})$; long-dash curve: $0.5 \odot \cos(10n + \frac{\pi}{3})$; solid curve: $trop(x[n])$.

In particular all terms but the first one have lower amplitudes. Hence the minimums of $trop(x[n])$, in this case, are the minimums of $0.5 \odot \cos(2.5n + \frac{\pi}{4})$, $0.5 \odot \cos(4.2n + \frac{\pi}{2})$ and $0.5 \odot \cos(10n + \frac{\pi}{3})$, as we can see in Figure 4.

In the two Figures we can distinguish also a different behavior about local minimums (i.e. points in which the first derivative is zero, but we do not get the lower possible value -0.5): in Figure 3 there are many local minimums given by the contributions of the terms different from $0.5 \odot \cos(2.5n + \frac{\pi}{4})$, which are not present in Figure 4. Notice that in both cases the term $1 \odot \cos(n)$ never contributes to the minimum values of $trop(x[n])$ since its amplitude (hence the translation of the term up in the plane) is too big with respect to the others.

In general, the analysis of the minimums is, at a first glance, quite hard since the number of minimums does not depend only on the amplitudes but also on the angular frequencies and phases.

The situation is better for a harmonic sound

$$x[n] = a_1 \cos(\omega n + \phi_1) + a_2 \cos(2\omega n + \phi_2) + \cdots + a_p \cos(p\omega n + \phi_p). \tag{3.3}$$

and, in particular, for the case when all sine waves have the same amplitude and zero phase, that is $a_1 = \cdots = a_p = \alpha$ and $\phi_1 = \cdots = \phi_p = 0$. Here all cosine terms reach the value 1 at the period $\frac{2\pi}{\omega R}$ of $\cos(\omega n)$ (where R is the sample rate). Hence in $\frac{2\pi}{\omega R}$, by Proposition 2, $trop(x[n])$ is at its maximum $1 + \alpha$.

For the minimum, we know, by Proposition 2, that is equal to $-1 + \alpha$. Since $cos(i\omega n)$ reaches, in one cycle, the minimum i times (with respect to $cos(\omega n)$) we can expect that the total number of minimums of $trop(x[n])$, in a period, is bounded by $\sum_{i=1}^{p} i = \frac{p(p+1)}{2}$ (remember that all amplitudes are equal). However some minimums are the same for different terms $cos(i\omega n)$ and $cos(k\omega n)$. As a matter of fact, if $\cos(i\omega n) = 0$ then $i\omega n = \pi$, which implies $k\omega n = \frac{k}{i}\pi$ giving again $\cos(k\omega n) = 0$ if $\frac{k}{i}$ is an odd integer number. Hence, in a cycle of $\cos(i\omega n)$ there are $\lfloor \frac{p}{2} - \frac{1}{2} \rfloor$ higher term sharing the same minimum of $\cos(i\omega n)$. Since $\cos(i\omega n)$

repeats i times during a cycle of $\cos(\omega n)$, the total number of minimums of $\cos(i\omega n)$, in common with higher terms, is $i\lfloor \frac{p}{2} - \frac{1}{2} \rfloor$.

For example, if $p = 3$ we get one common minimum between $cos(\omega n)$ and $cos(3\omega n)$ hence the total number of minimums of $trop(x[n])$ is 5, as shown in Figure 5 (here $\frac{p(p+1)}{2} = \frac{3 \cdot 4}{2} = 6$). If $p = 5$, the total number of minimums of $trop(x[n])$ is 13 (Figure 6). Since $\frac{5 \cdot 6}{2} = 15$ this means there are two common minimums which are the ones shared by $\cos(\omega n)$ with $\cos(3\omega n)$ and $\cos(5\omega n)$. If $p = 10$ the total number of minimums of $trop(x[n])$ is 45 (Figure 7). Since $\frac{10 \cdot 11}{2} = 55$, this means there are ten common minimums: 4 minimums shared by $\cos(\omega n)$ with $\cos(3\omega n)$, $\cos(5\omega n)$, $\cos(7\omega n)$ and $\cos(9\omega n)$, 4 minimums (two for half period) shared by $\cos(2\omega n)$ with $\cos(6\omega n)$ and $\cos(10\omega n)$ and finally 3 minimums (one for one third of the period) shared by $\cos(3\omega n)$ with $\cos(9\omega n)$, but, since the central one is already included in the first case, here we count only two minimums.

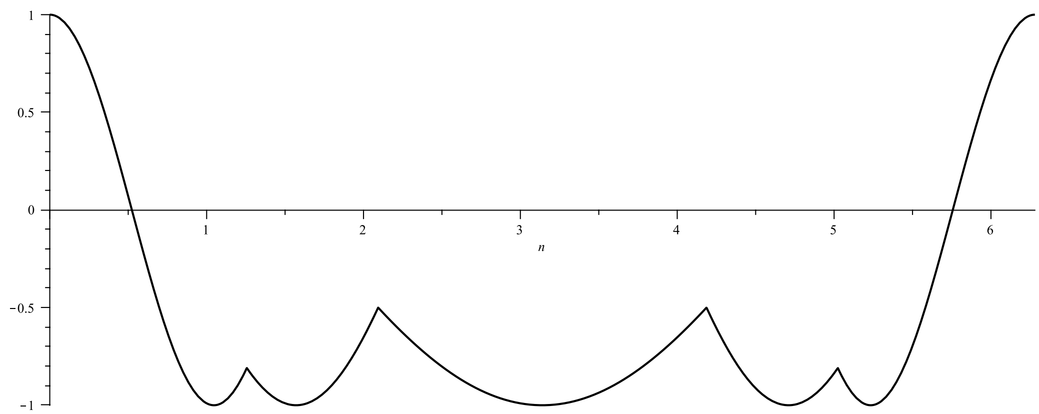

FIGURE 5. Minimums of $trop(x[n])$ with $p = 3$, $a_1 = \cdots = a_p = \alpha$ and $\phi_1 = \cdots = \phi_p = 0$.

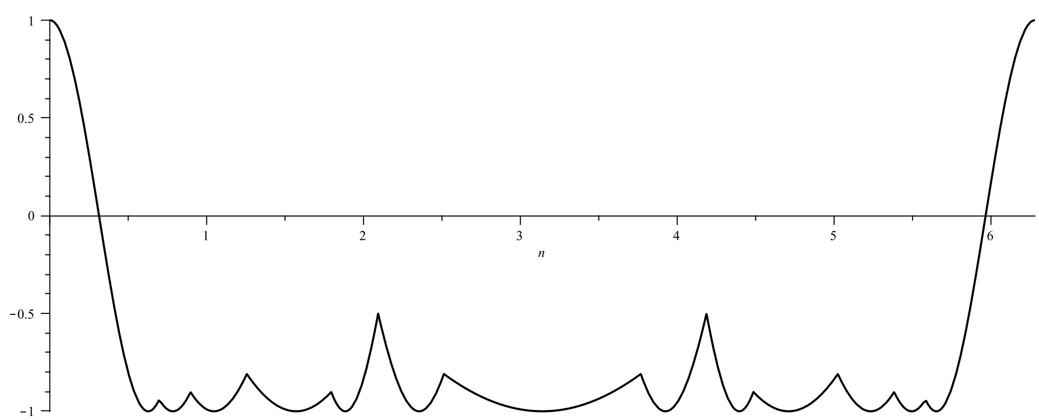

FIGURE 6. Minimums of $trop(x[n])$ with $p = 5$, $a_1 = \cdots = a_p = \alpha$ and $\phi_1 = \cdots = \phi_p = 0$.

We know that using odd or even harmonics produces different sound results. However, from the definition of tropical multiplication, we know that in our system we cannot control the partials amplitudes. Hence

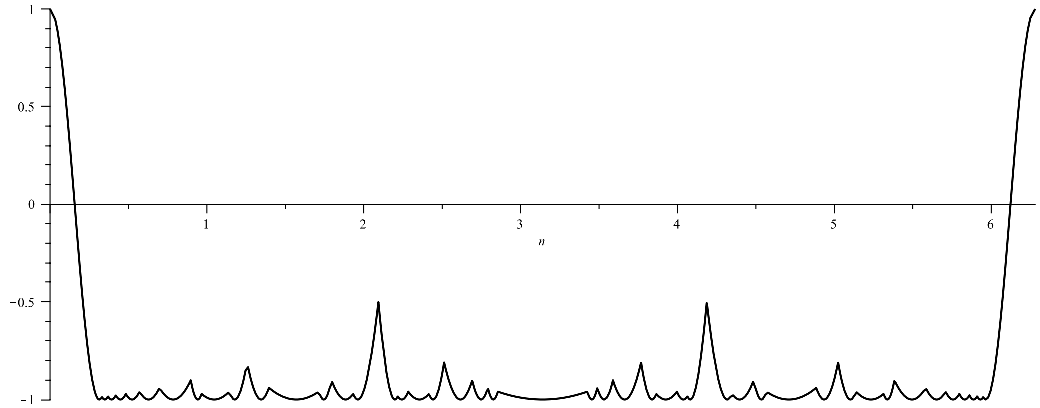

FIGURE 7. Minimums of $trop(x[n])$ with $p = 10$, $a_1 = \cdots = a_p = \alpha$ and $\phi_1 = \cdots = \phi_p = 0$.

we can at least take in consideration the possibility to use all the partials or the odd or the even ones by eliminating one or another from the original Additive Equation (3.1). This means, tropically speaking, to set to ∞ the amplitudes of unwanted oscillators; as a matter of fact a term $\infty \odot \cos(\omega_i n + \phi)$ is always equal to ∞, by Formula (2.1), and this term will be bigger than other term in (3.2), so without giving any contribution. Hence given $x[n]$ as in (3.1), we can define the following three kinds of tropicalization:

$$\begin{array}{ll} \text{full mode} & a_1 \odot \cos(\omega_1 n + \phi_1) \oplus a_2 \odot \cos(\omega_2 n + \phi_2) \oplus a_3 \odot \cos(\omega_3 n + \phi_3) \oplus \cdots \\ \text{even mode} & a_1 \odot \cos(\omega_1 n + \phi_1) \oplus a_2 \odot \cos(\omega_2 n + \phi_2) \oplus a_4 \odot \cos(\omega_4 n + \phi_4) \oplus \cdots \\ \text{odd mode} & a_1 \odot \cos(\omega_1 n + \phi_1) \oplus a_3 \odot \cos(\omega_3 n + \phi_3) \oplus a_5 \odot \cos(\omega_5 n + \phi_5) \oplus \cdots \end{array} \tag{3.4}$$

All the facts stated until now were related to the tropicalization full mode, but they apply in a similar way to the other tropicalizations in even mode or odd mode.

Remark 4. Let us end this section with a first important remark about tropical additive synthesis. From the graph of Figure 3 we can observe that the waveform $trop(x[n])$ can be also constructed as a piecewise curvilinear function. However, such construction, for example using wavetable synthesis, would require, as the number of sine terms in $trop(x[n])$ grows, an increasing and massive number of data.

3.1. **Tropical classical waveforms.** It is known that every periodic waveform can be approximated by a finite sum of sine wave and this approximation will be better as the number of addends grows. Following this idea we will write the approximations of sawtooth, square and triangle waves and then we will tropicalize them.

The approximation of the sawtooth $s[n]$ (Figure 8, on the left) can be written as

$$s[n] = a_1 \cos(\omega n + \frac{\pi}{2}) + a_2 \cos(2\omega n + \frac{\pi}{2}) + \cdots + a_p \cos(p\omega n + \frac{\pi}{2})$$

where $a_i = \frac{1}{i}$ for all i. The tropicalization of the sawtooth (Figure 8 on the right) is then

$$trop(s[n]) = 1 \odot \cos(\omega n + \frac{\pi}{2}) \oplus \frac{1}{2} \odot (2\omega n + \frac{\pi}{2}) \oplus \cdots \oplus \frac{1}{p} \odot \cos(p\omega n + \frac{\pi}{2})$$

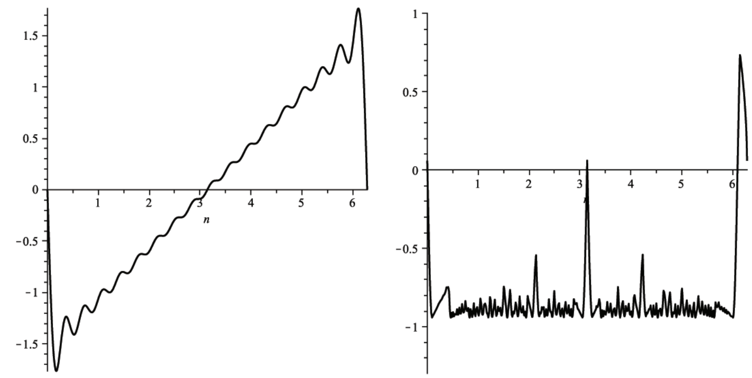

FIGURE 8. Approximation of sawtooth and its tropicalization, $p = 17$.

The approximation of the square waveform $sq[n]$ (Figure 9 on the left) can be written as

$$sq[n] = a_1 \cos(\omega n + \frac{\pi}{2}) + a_2 \cos(2\omega n + \frac{\pi}{2}) + \cdots + a_p \cos(p\omega n + \frac{\pi}{2})$$

where

$$a_i = \begin{cases} \frac{1}{i} & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases}.$$

Finally the approximation of the triangle waveform $t[n]$ (Figure 10 on the left) can be written as

$$t[n] = a_1 \cos(\omega n) + a_2 \cos(2\omega n) + \cdots + a_p \cos(p\omega n)$$

where

$$a_i = \begin{cases} \frac{1}{i^2} & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases}.$$

Before to tropicalize square and triangle waveforms a consideration is mandatory. For the way we defined the Fourier series of these waveforms, every even harmonic has zero amplitude, so a term $0 \cdot \cos(2i\omega n + \phi)$ is just 0 and in classical mathematics we would just ignore such terms. However, in tropical mathematics, 0 is the neutral element of tropical multiplication and $0 \odot \cos(2i\omega n + \phi)$ is not zero but $0 + \cos(2i\omega n + \phi)$ which is just $\cos(2i\omega n + \phi)$. Hence, we can tropicalize in two different ways: a full mode tropicalization, considering all even harmonics in the form $0 \cdot \cos(2i\omega n + \phi)$, or an odd mode tropicalization, which is equivalent to tropicalize the original Fourier series once we removed the even harmonics.

Hence the tropicalization of the square waveform, full mode, (Figure 9 on the middle) is

$$trop(sq[n]) = 1 \odot \cos(\omega n + \frac{\pi}{2}) \oplus (2\omega n + \frac{\pi}{2}) \oplus \frac{1}{3} \odot \cos(3\omega n + \frac{\pi}{2}) \oplus (4\omega n + \frac{\pi}{2}) \oplus \cdots$$

while the tropicalization of the square waveform, odd mode, (Figure 9 on the right) is

$$trop(sq[n]) = 1 \odot \cos(\omega n + \frac{\pi}{2}) \oplus \frac{1}{3} \odot \cos(3\omega n + \frac{\pi}{2}) \oplus \cdots$$

Similarly the tropicalization of the triangle waveform, full mode, (Figure 10 on the middle) is

$$trop(t[n]) = 1 \odot \cos(\omega n) \oplus (2\omega n) \oplus \frac{1}{9} \odot \cos(3\omega n) \oplus (4\omega n) \oplus \cdots$$

while the tropicalization of the triangle waveform, odd mode, (Figure 10 on the right) is

$$trop(t[n]) = 1 \odot \cos(\omega n) \oplus \frac{1}{9} \odot \cos(3\omega n) \oplus \cdots$$

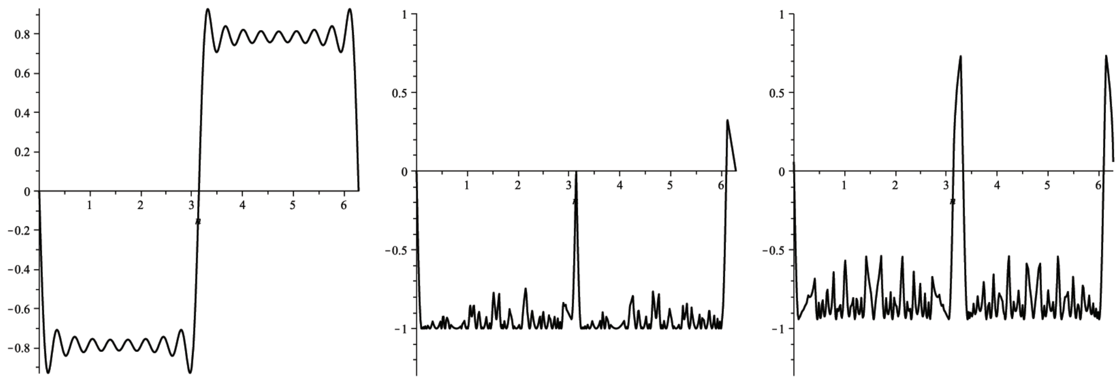

FIGURE 9. Approximation of square waveform and its tropicalizations, $p = 17$.

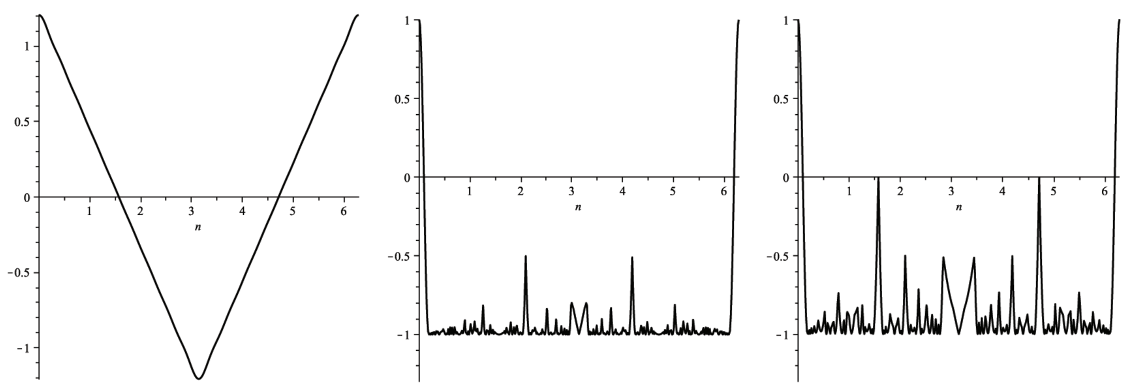

FIGURE 10. Approximation of triangle waveform and its tropicalizations, $p = 17$.

3.2. **Tropical Beatings.** A beating, in classical additive synthesis, is the physical effect obtained summing two, or more, sine waves with angular frequencies that differ slightly from each other. The two waves start out in phase, with the resulting constructive interference causing the overall amplitude to grow, but after a certain number of oscillations, they gradually shift to being maximally out of phase with each other, which causes the overall amplitudes to be reduced. After a certain number of oscillations, the waves are once again in phase, and the cycle repeats ([1]).

If we consider the sum of two waves $b[n] = a_1 \cos(\omega_1 n) + a_2 \cos(\omega_2 n)$, where ω_1 and ω_2 are slightly different, its tropicalization is simply

$$trop(b[n]) = a_1 \odot \cos(\omega_1 n) \oplus a_2 \odot \cos(\omega_2 n) = \min\{a_1 + \cos(\omega_1 n), a_2 + \cos(\omega_2 n)\}.$$

The cyclic constructive and disruptive interferences of $a_1 \cos(\omega_1 n)$ and $a_2 \cos(\omega_2 n)$ give rise to a similar cyclic event in the tropicalization: constructive interferences correspond to points in which $trop(b[n])$ is at its maximum $1 + \min\{a_1, a_2\}$, while disruptive interference correspond to points in which $trop(b[n])$ is at the minimum at $-1 + a_1$ or $-1 + a_2$. Even though such minimums are achieved in many other points, these constructive and disruptive interferences give rise to a periodic form of the amplitude envelope. Hence, also for $trop(b[n])$ we get an effect of beating, that we call *tropical beating*.

Figure 11 compares beatings and tropical beatings for two waves with angular frequencies 1 and 1.1, where we change the amplitude of the second waves.

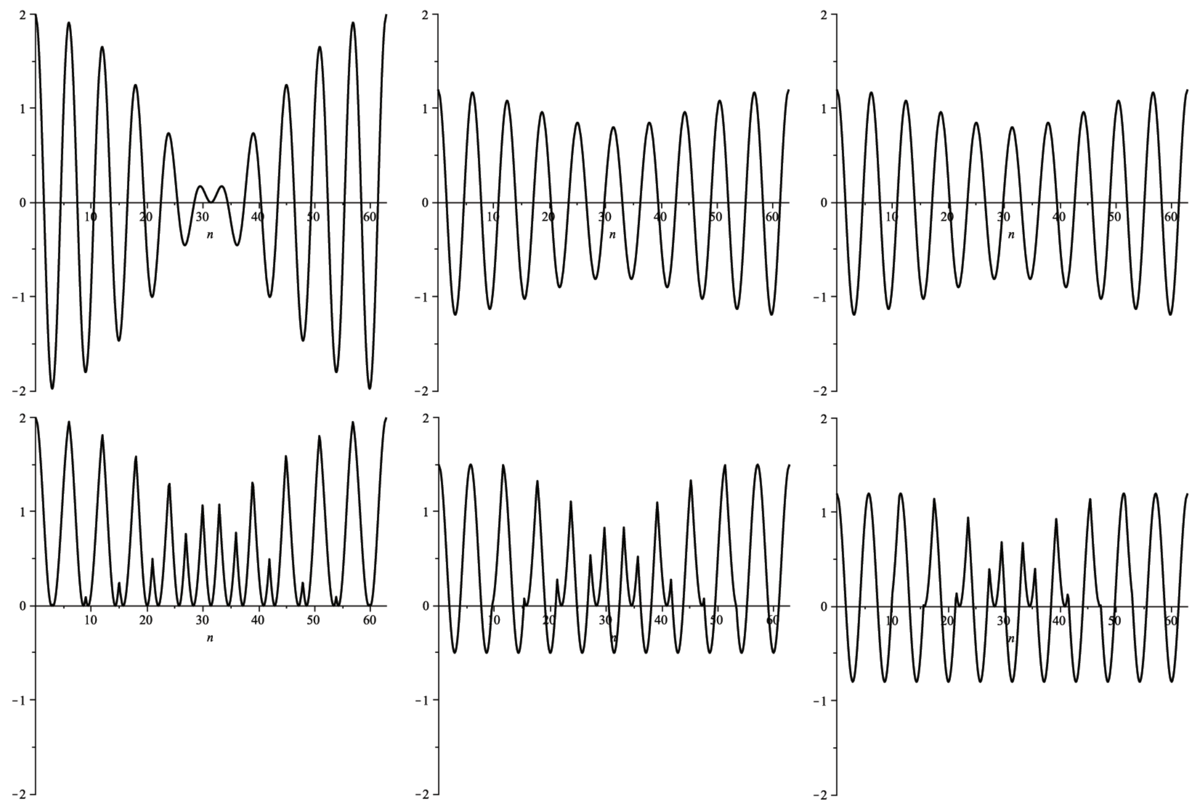

FIGURE 11. Top: Beatings of $1 \cdot \cos(n) + a \cdot \cos(1.1n)$; bottom: tropical beatings of $1 \odot \cos(n) \oplus a \odot \cos(1.1n)$. Here a is respectively 1, 0.5 and 0.2.

4. Practical approaches, the Tropical Oscillator or $Trop(x[n])$-oscillator

After the matemathical definition, it comes the *mise in oeuvre* of a coherent and useful musical tool. While the mathematician formalizes an idea exploring mathematics itself, the musician holds the responsibility of concretizing the theory in a musical way, in order to realize the architecture of what we call a *Tropical Oscillator* or $Trop(x[n])-oscillator$. Before describing the oscillator let us make, first, few remarks on some key differences between Classic Additive Synthesis and the Tropical Additive Synthesis counterpart.

In the Tropical realm all the oscillators have the same amplitude. This fact occurs because the tropical VCA is a sum and not a multiplication, hence "tropical amplitude" means that we basically add a 0Hz signal. In practice the tropical VCA shifts the waveform up and down, in positive or negative values, while the waveform's amplitude is left unchanged. We must therefore take in consideration that using a new algebra implies that we should use a number of tailored approaches to Additive Synthesis.

The number of produced harmonics is much higher than the number of cosines used for the equation. The reader have noticed that the minimum function, instead of adding the waves, makes them interact and create complex envelopes. The spectral shape grows immediately, producing also aliasing effects, which interact positively and destructively with the already crowded spectra; therefore in Tropical Additive Synthesis even two oscillators alone can create a highly rich sound. As pointed out in Remark 4, we must also consider that to build a tropical wave by means of curved functions it would mean have a growing and massive number of data to write each time we add one oscillator to the equation. This should clarifies why the Tropical Oscillator is an efficient timbre generator.

5. $Trop(x[n])$ Architecture

While two oscillators give interesting results, even more complex and varied sounds can be obtained by adding few more oscillators in the design.

Let's say that the ideal number of generators in the $Trop(x[n])-$oscillator is five. The design of the oscillator should therefore include switches to control the principal (or fundamental) oscillator and the four harmonic (or subharmonic) oscillators (Figure 12).

Master Tuner. The principal oscillator can be tuned with a master tuning control and include also a frequency multiplier that acts as an octave selector or, as an arbitrary multiplier.

Harmonics' Multipliers. Each harmonic oscillator has also a selectable frequency multiplier to select the order of the harmonics (or sub-harmonics) used by the each generator. In this way a great number of choices is available at the musician with odd or even harmonics or eventually with random numbers to create various shapes of noise.

Detuners. Depending on the user interface we choose to build around our instrument we could have full control to each single oscillator's frequency, or better, a master tune control and four fine-tune detuners: $a_1 \odot \cos(\omega_1 n + \phi_1) \oplus a_2 \odot \cos((\omega_2 + \delta_2)n + \phi_2) \oplus \oplus a_3 \odot \cos((\omega_3 + \delta_3)n + \phi_3) \cdots$. Detuners, which add or subtract a few Hz from each function, are also used to create Tropical Beatings.

Tropical VCAs. Every oscillator has a Tropical VCA that sums a value between -1 and 1 to the incoming signal. The combination of these five values, along with the harmonic multipliers can shape the timbre into a great number of possible waveforms.

VCAs Modulation. Interesting harmonic effects emerge if we modulate all the Tropical VCAs at once with a single cosine function using a different phase σ_i for each VCA: $\cos(\tilde{\omega} n + \sigma_1) \odot \cos(\omega_1 + \phi_1) \oplus \cos(\tilde{\omega} n +$

$\sigma_2) \odot \cos(\omega_2 n + \phi_2) \oplus \cdots$. This is easily obtained using a serie of delays. Also the use of multiple modulators, slightly detuned between each other, can be a solution.

DC offset or another Tropical VCA. As pointed out in Remark 3, using the Tropical VCAs we can expect to have a shift of the waveforms over or below 0, therefore a final Tropical VCA is introduced to correct the DC offset.

6. Oscillators' examples

Even though it is not possible to describe here the infinite audio possibilities of the $Trop(x[n])-$oscillator, we end this paper showing few examples, where we can notice how the waveforms change according to the modes and the tropical VCA's.

The examples in Figures 13, 14 and 15 are in full mode (odd+ even harmonics). The examples in Figures 16, 17 and 18 are in odd mode (odd harmonics only). The examples in Figures 19, 20 and 21 are in even mode (even harmonics only). Two periods of the waveforms are showed. Notice that, since the phases are all zero, the waveforms are symmetric with respect to half period.

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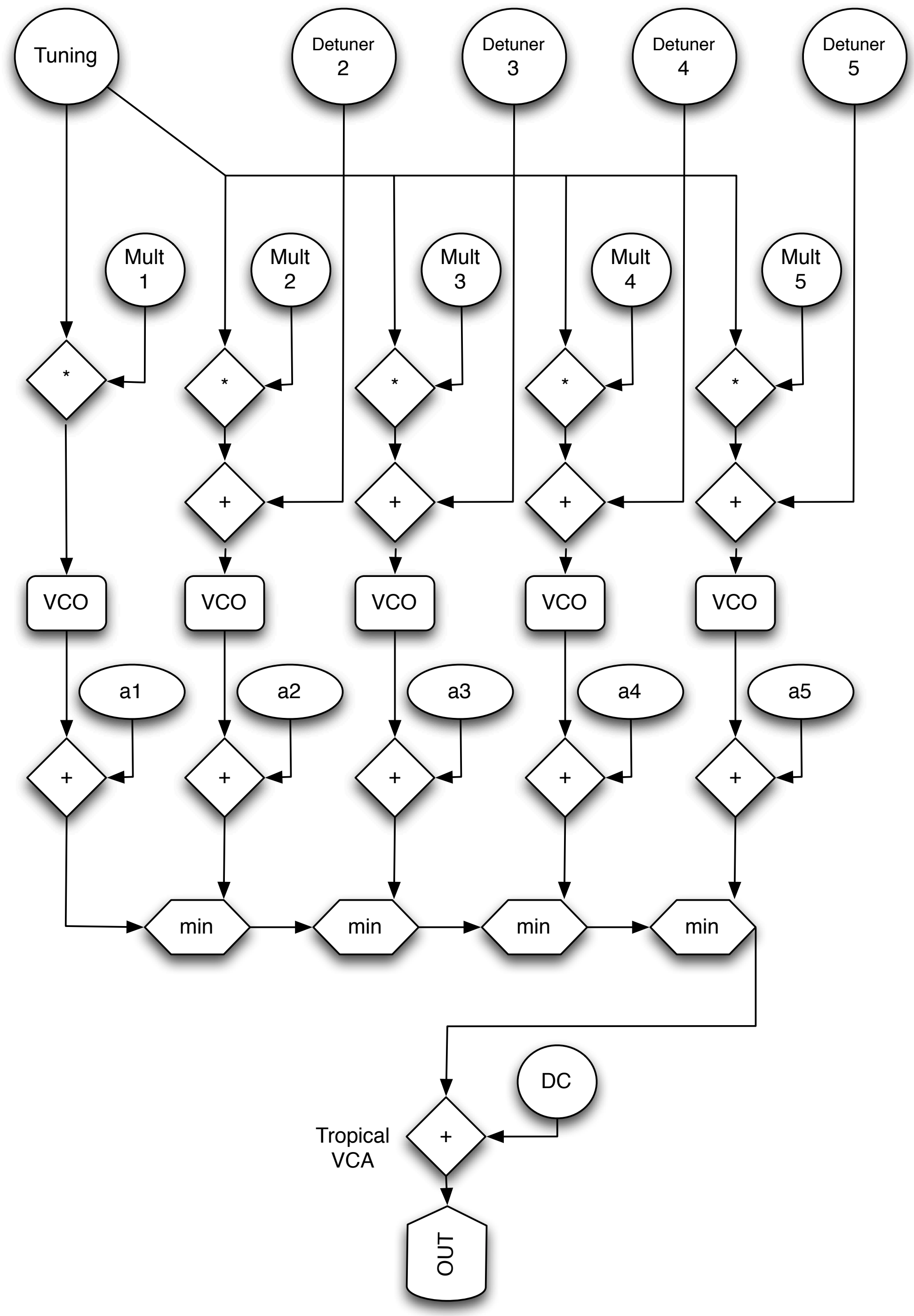

FIGURE 12. The technical scheme of the Tropical Oscillator.

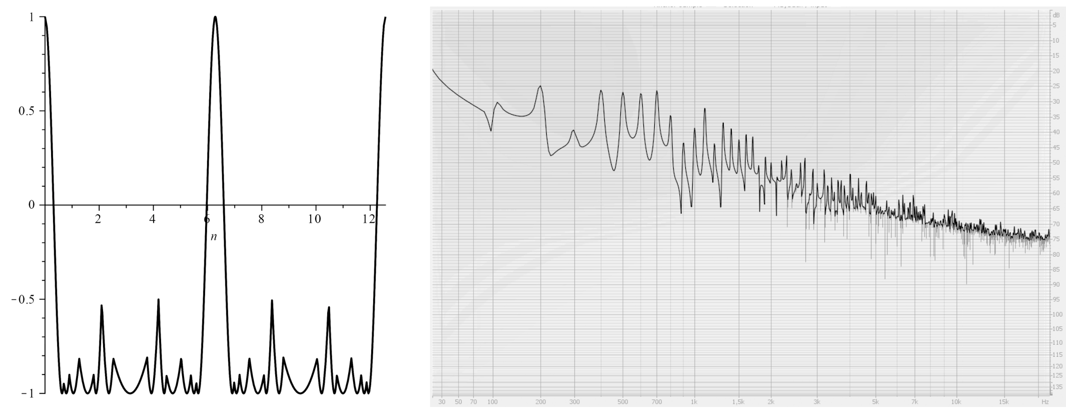

FIGURE 13. $0 \odot \cos(n) \oplus 0 \odot \cos(2n) \oplus 0 \odot \cos(3n) \oplus 0 \odot \cos(4n) \oplus 0 \odot \cos(5n)$.

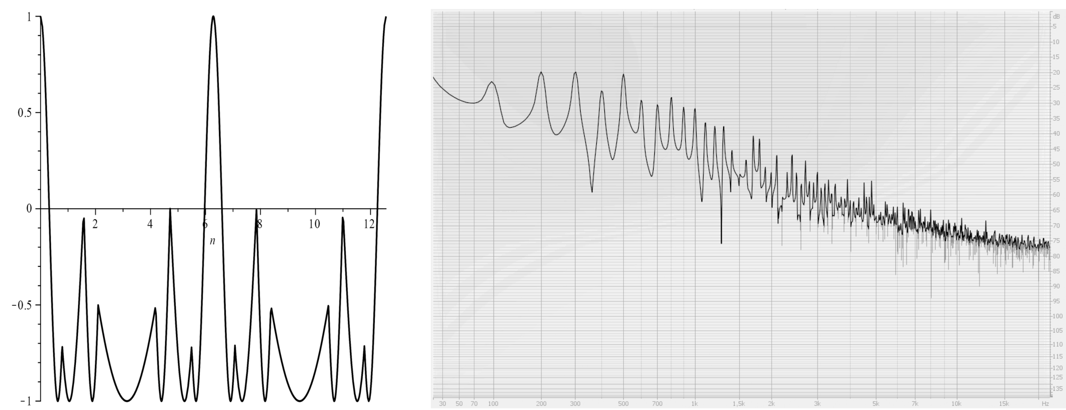

FIGURE 14. $-1 \odot \cos(n) \oplus 1 \odot \cos(2n) \oplus -1 \odot \cos(3n) \oplus 1 \odot \cos(4n) \oplus -1 \odot \cos(5n) \oplus 1$.

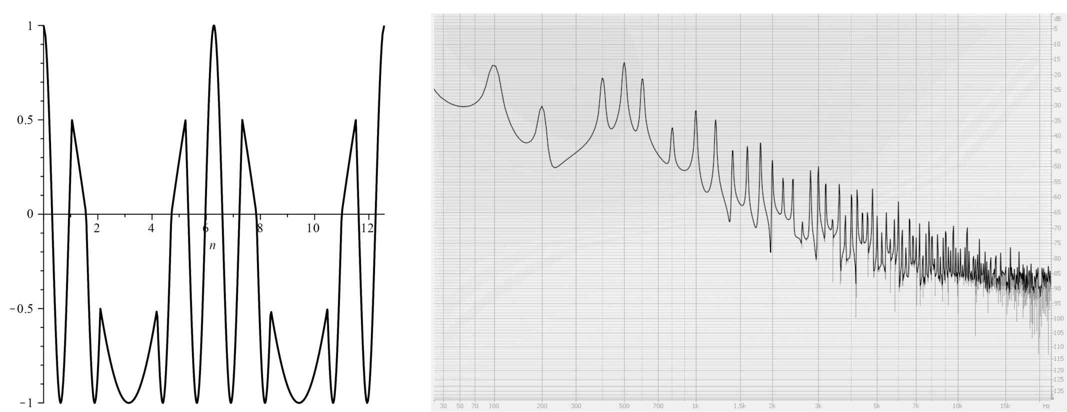

FIGURE 15. $-1 \odot \cos(n) \oplus 1 \odot \cos(2n) \oplus 1 \odot \cos(3n) \oplus 1 \odot \cos(4n) \oplus -1 \odot \cos(5n) \oplus 1$.

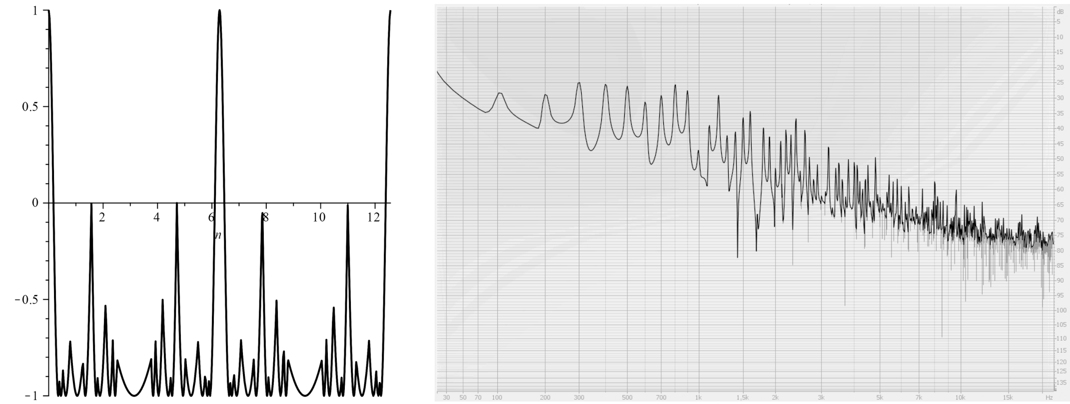

FIGURE 16. $0 \odot \cos(n) \oplus 0 \odot \cos(3n) \oplus 0 \odot \cos(5n) \oplus 0 \odot \cos(7n) \oplus 0 \odot \cos(9n)$.

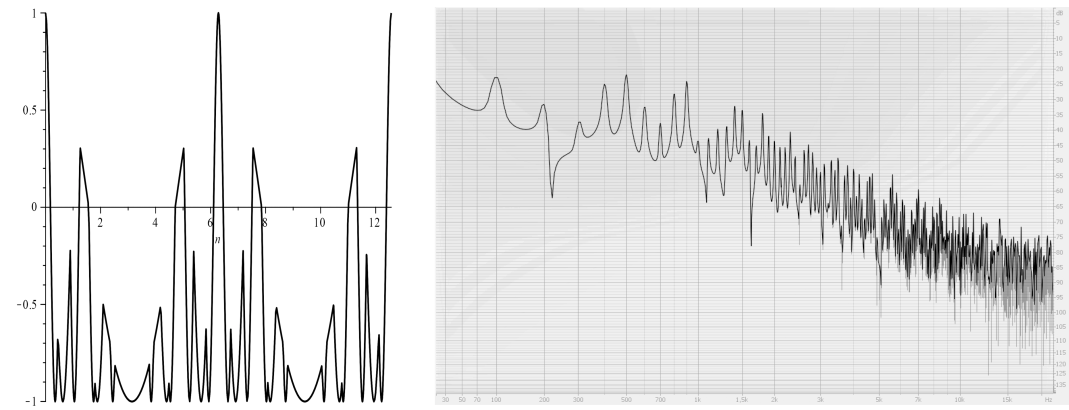

FIGURE 17. $-1 \odot \cos(n) \oplus 1 \odot \cos(3n) \oplus -1 \odot \cos(5n) \oplus 1 \odot \cos(7n) \oplus -1 \odot \cos(9n) \oplus 1$.

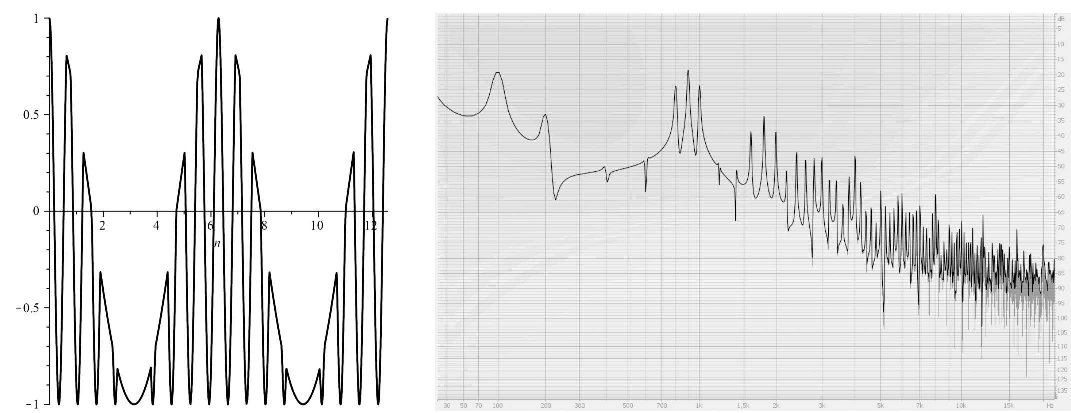

FIGURE 18. $-1 \odot \cos(n) \oplus 1 \odot \cos(3n) \oplus 1 \odot \cos(5n) \oplus 1 \odot \cos(7n) \oplus -1 \odot \cos(9n) \oplus 1$.

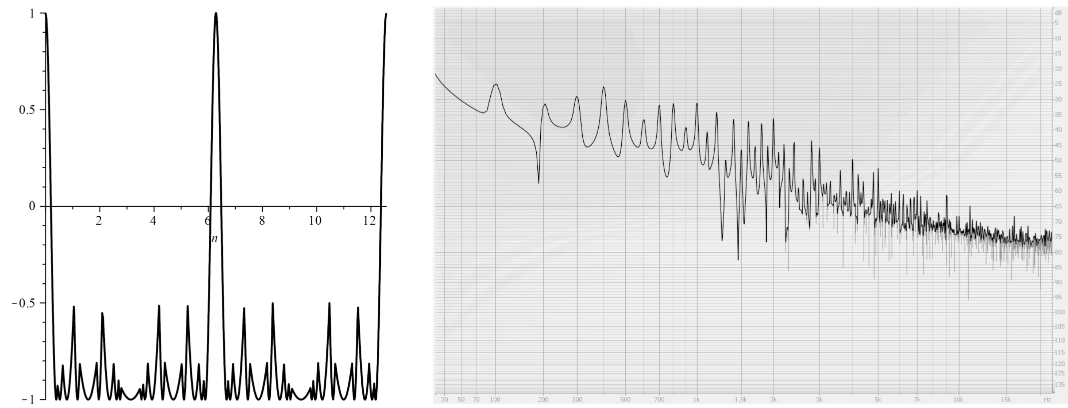

FIGURE 19. $0 \odot \cos(n) \oplus 0 \odot \cos(2n) \oplus 0 \odot \cos(4n) \oplus 0 \odot \cos(6n) \oplus 0 \odot \cos(8n)$.

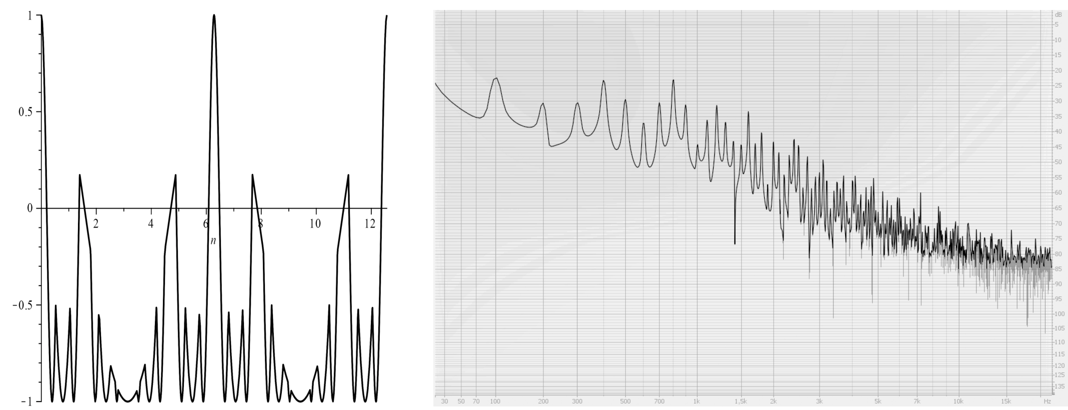

FIGURE 20. $-1 \odot \cos(n) \oplus 1 \odot \cos(2n) \oplus -1 \odot \cos(4n) \oplus 1 \odot \cos(6n) \oplus -1 \odot \cos(8n) \oplus 1$.

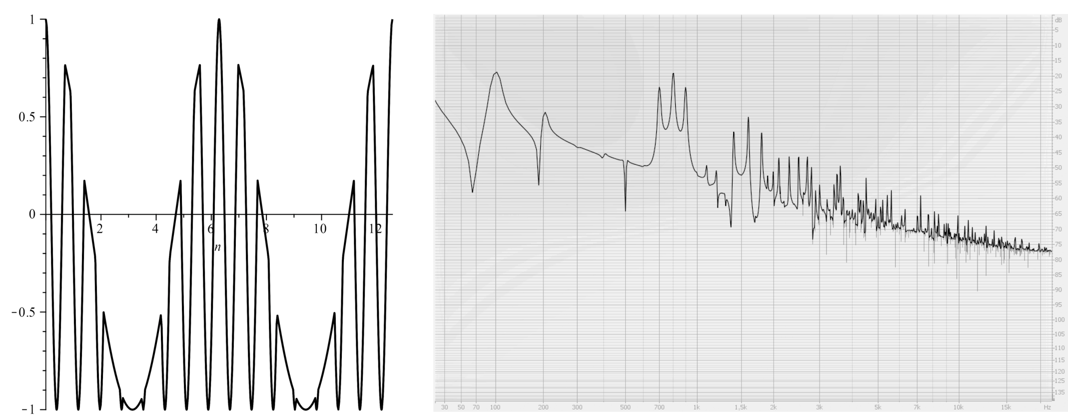

FIGURE 21. $-1 \odot \cos(n) \oplus 1 \odot \cos(2n) \oplus 1 \odot \cos(4n) \oplus 1 \odot \cos(6n) \oplus -1 \odot \cos(8n) \oplus 1$.